A construction of G_2 holonomy spaces with torus symmetry.

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Abstract

In the present work the Calderbank-Pedersen description of four dimensional manifolds with self-dual Weyl tensor is used to obtain examples of quaternionic-kahler metrics with two commuting isometries. The eigenfunctions of the hyperbolic laplacian are found by use of Backglund transformations acting over solutions of the Ward monopole equation. The Bryant-Salamon construction of G_2 holonomy metrics arising as R^3 bundles over quaternionic-kahler base spaces is applied to this examples to find internal spaces of the M-theory that leads to an $N = 1$ supersymmetry in four dimensions. Type IIA solutions will be obtained too by reduction along one of the isometries. The torus symmetry of the base spaces is extended to the total ones.

1. Introduction and the main result.

The classification of the possible holonomy groups of Riemannian or Pseudo-Riemannian manifolds is an old mathematical problem. In [1] Berger presented a list of the possible restricted holonomy groups of N-dimensional Riemannian manifolds, but after the completion of this work it remained to prove the existence of metrics with exceptional holonomies G_2 and $Spin(7)$ for the seven and eight dimensional cases respectively. This was achieved successfully by Bryant in [2]. In general, if a given Riemannian metric with dimension N admits at least one covariantly constant spinor satisfying $D_i\eta = 0$ the holonomy group will be $SU(\frac{n}{2})$, $Sp(\frac{n}{4})$, G_2 or $Spin(7)$, the last two cases corresponding to seven and eight dimensions. For G_2 holonomy manifolds there is exactly one. The fact that this spinor exist is apparent from the decomposition $8 = 1 \oplus 7$ of the spinor representation of the tangent space $SO(7)$. Equivalently, in such manifolds one can choose an orthogonal frame e^i in which the three octonionic form

$$\Phi = e^1 \wedge e^2 \wedge e^7 + e^1 \wedge e^3 \wedge e^6 + e^1 \wedge e^4 \wedge e^5 + e^2 \wedge e^3 \wedge e^5 + e^4 \wedge e^2 \wedge e^6$$

$$+ e^3 \wedge e^4 \wedge e^7 + e^5 \wedge e^6 \wedge e^7$$

and its dual $*\Phi$ are closed [3].

Many G_2 holonomy metrics are known (see for instance [4], [5], [6] and [7]). All of them have vanishing Ricci tensor, i.e, they are Ricci-flat, and this implies that they are vacuum solutions of the Einstein equation. This suggest that the construction of G_2 holonomy manifolds is related

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with the construction of seven dimensional self-dual manifolds. The self-duality condition in seven dimensions

$$R_{ab} = \pm\frac{c_{abcd}}{2}R_{cd}$$

implies Ricci-flatness and G_2 restricted holonomy [10] (analogous considerations hold in eight dimensions for $Spin(7)$). Given an anzatz for a metric, this condition gives a system of equations to be satisfied in order to have a G_2 manifold. Although this equations are non-linear they have been solved in cases with suitable symmetries in which the system takes a more simple form [6].

The main feature that relates this subject to physics is the presence of one nonzero parallel spinor field η, which plays a central role in supersymmetry, string theory and M-theory. Compactification of the M-theory (or his low energy limit, the eleven dimensional supergravity) on G_2 holonomy manifolds leads to an effective four dimensional theory with one supersymmetry, corresponding to such spinor field [8].

There are in the literature examples of "weak G_2 holonomy" [13], which are again backgrounds of the M-theory that give rise to N=1 supersymmetry in $D = 4$. In this case, there is an spinor field η which is not covariantly constant but satisfies $D_i\eta \sim \lambda\gamma_i\eta$. The Ricci flatness condition is replaced by $R_{ij} \sim \lambda g_{ij}$. In the limit $\lambda \to 0$ one obtain G_2 as restricted holonomy group. Hitchin has shown that under certain conditions is possible to construct this kind of manifolds starting with an $Spin(7)$ holonomy one [9].

Compactification based on G_2 smooth manifolds cannot give rise to chiral matter. In the smooth case the harmonic Kaluza-Klein decomposition of the eleven-dimensional supergravity is the N=1 four dimensional supergravity coupled Abelian vector multiplets plus chiral multiplets. But the chiral matter fields can emerge only if the manifold develops a singularity, as pointed out in [15]. It turns out that to obtain a realistic model one should investigate the dynamics of the M-theory over orbifolds. A modern description of this dynamics over manifolds that are developing a conical singularities can be found in [14].

The study of explicit metrics with exceptional holonomy has also importance in the context of dualities of string theory and M theory ([12],[16],[18] and [19]). The range of applications of this topic is very wide; some of them can be found in [21]-[27].

Certain G_2 metrics with two abelian isometries have called the attention recently [34], because an $U(1)$ isometry allows an type IIA superstring interpretation upon dimensional reduction to ten dimensions. In [43] it has been extracted the IIA reduction of M-theory on certain toric backgrounds, and its type IIB duals. Such IIA solutions corresponds to systems of weakly and strongly coupled D6-branes, while the duals describes systems of localized and delocalized 5-branes. The G_2 spaces presented in those works have been found applying the Bryant-Salamon construction with a four dimensional quaternionic base manifold with torus symmetry [33]; the $U(1) \times U(1)$ isometry is extended to the total space. In the hyperkahler limit this construction gives G_2 holonomy spaces which are globally the cartesian product of the hyperkahler one with R^3.

One of the purposes of this paper is to describe those toric base manifolds. The construction of hyperkahler ones is reduced to find Axisymmetric Harmonic Functions (AHF), which are solutions of the "monopole equation"

$$V_{\eta\eta} + \rho^{-1}(\rho V_\rho)_\rho = 0$$

described by Ward and Woodhouse in [35]. By another side, all the quaternionic metrics with $U(1) \times U(1)$ isometry have been classified recently by Calderbank and Pedersen in [38]; their method is associated to solve an equation that is related to the monopole one by a Backglund transformation. The equation is

$$F_{\rho\rho} + F_{\eta\eta} = \frac{3F}{4\rho^2}$$

and is seen that, by definition, its solutions F are eigenfunctions of the two dimensional laplacian operator with eigenvalue 3/4. By use of Backglund mappings non trivial examples of quaternionic kahler metrics will be found, together with the G_2 metrics arising as an R^3 bundle over such spaces.

The organization of this paper is as follows: in section 2 it is presented the group G_2 as the group of automorphisms of the octonions. In section 3 there are reviewed some basic features about self-dual seven dimensional manifolds. In section 4 it is shown that such manifolds have restricted holonomy group G_2. Section 5 contains the main discussion of this work. Examples of quaternionic kahler and hyperkahler metrics are presented and the Bryant-Salamon construction is applied to such examples to find G_2 metrics. The vacuum configurations of to the eleven dimensional supergravity related to this metrics are found. Dimensional reduction of such configurations alone one of the isometries gives rise to ten dimensional type IIA backgrounds. The author hopes that the first four sections will give to the reader new in this area a clear (and elementary) idea about the main features of G_2 holonomy spaces.

2. The exceptional group G_2 and the octonions.

The group $G_2 \subset SO(7)$ is one of the exceptional simple Lie groups. It is compact, connected, simply-connected and of dimension 14. It has been proved [28], that G_2 is the group of automorphisms of the octonions O, up to an isomorphism. The octonions (or Cayley numbers) constitutes the only non associative division algebra (the associative ones are only R, C and H) and an arbitrary element $x \in O$ can be written as a linear combination of the form $\overrightarrow{x} = x^0 + x^i e_i$, where the set e_i constitute a basis of 7 unit octonions with the following multiplication rule:

$$e_i e_j = c_{ijk} e_k; \quad e_i.1 = 1.e_i = e_i \tag{2.1}$$

The x^i's takes real values. The subspace P of O generated by the elements $x = x^i e_i$ is called the space of "pure octonions", and the total space can be decomposed as $O = R \bigoplus P$. The constants c_{ijk} that define the multiplication (2.1) are totally antisymmetric and

$$c_{123} = c_{246} = c_{435} = c_{367} = c_{651} = c_{572} = c_{714} = 1, \tag{2.2}$$

up to an index permutation. The constants corresponding to another set of indices are identically zero. From (2.1) and (2.2) it is seen that $(e^3 e^7)e^5 - e^3(e^7 e^5) = -e^1$, which shows the non-associativity of the octonion algebra. For this reason the octonions cannot be represented as a matrices and do not satisfy the Jacobi identities. In other words, the algebra of O is not a Lie algebra.

It is possible to represent the components of an arbitrary octonion as a 7-dimensional vector $\overrightarrow{X} = (x_1, ..., x_7)$. We define the octonion numbers $g\overrightarrow{x}$ as those with components $g.\overrightarrow{X}$, where g is an arbitrary 7×7 matrix. The statement that the exceptional group G_2 is the group of automorphism of the octonions means if $x \in P$, $gx \in P$ if g is any of the elements of G_2 in the fundamental irreducible representation; and that if $x.y = z$ for given x, y and z belonging to P, $gx.gy = gz$.

Over O it is defined an internal product $(,) : O \times O \to R$ given by

$$(e_i, e_j) = \delta_{ij} \ , \tag{2.3}$$

from where it is obtained that

$$(x, y) = x^i y^i \ . \tag{2.4}$$

Taking into account all the facts mentioned above it is possible to construct a three-form over a seven dimensional space V, which is G_2 invariant. This form is fundamental in this work because its closure has implications about the holonomy group of V. The construction follows decomposing the octonion space as $O = R \bigoplus P$, and defining over P the bilinear $B(x, y)$ and the internal product $<,>: P \times P \to R$ by the identities

$$(x^0 e_0 + x, y^0 e_0 + y) = x^0 y^0 + < x, y > , \tag{2.5}$$

$$(x^0 e_0 + x)(y^0 e_0 + y) = (x^0 y^0 - < x, y >) e_0 + (x^0 y + y^0 x + B(x, y)) \ . \tag{2.6}$$

Under an automorphism transformation $B(x, y)$ and $<,>$ satisfy

$$B(gx, gy) = gB(x, y) \ , \tag{2.7}$$

$$B(x, y) = -B(y, x) \ , \tag{2.8}$$

$$< gx, gy >=< x, y > \ . \tag{2.9}$$

From (2.7) and (2.9), it is seen that

$$\Phi(x, y, z) =< B(x, y), z >=< B(gx, gy), gz >= \Phi(gx, gy, gz) \ . \tag{2.10}$$

In other words, the trilinear $\Phi(x, y, z) =< B(x, y), z >$ is G_2 invariant. From the rule (2.1) and the definition (2.6) it follows that components of Φ are

$$\Phi(e_a, e_b, e_c) =< B(e_a, e_b), e_c >= c_{abc} \ ,$$

and that

$$\Phi(x, y, z) = c_{abc} x^a y^b z^c .$$

From the invariance of Φ under the action of G_2 follows the statement:

"For a given real vector space V of dimension 7, with $e_1, .., e_7$ a basis for V, the three form

$$\Phi(x, y, z) = \frac{1}{3!} c_{abc} e^a \wedge e^b \wedge e^c \tag{2.11}$$

is G_2 invariant."

3. Self-dual manifolds in 4 and 7 dimensions.

The self-duality condition is a familiar concept in quantum field theory [29] and in general relativity. By definition the curvature tensor $R_{ab} = dw_{ab} + w_{ac} \wedge w_{cb}$ of a Riemannian metric is self-dual if

$$R_{ab} = \frac{1}{2} \epsilon_{abmn} R_{mn}, \tag{3.12}$$

or, in components

$$R_{abcd} = \frac{1}{2}\epsilon_{abmn}R_{mncd}.$$

Many non trivial solutions of this type has been found in the past [30]. They are called "gravitational instantons" if their are Euclidean and with finite energy. It has been shown that in four dimensions a torsion-free metric that satisfies (3.12) will be a vacuum solution of the Einstein equations without cosmological constant [31].

In seven dimensions (3.12) is generalized as [10]

$$R_{ab} = \frac{1}{2}c_{abcd}R_{cd}\ . \tag{3.13}$$

where the totally antisymmetric c_{abcd} are defined in terms of the octonion structure constants (2.2) through the relations:

$$c_{abcd} = \frac{1}{3!}\epsilon^{abcdefg}c_{efg}\ . \tag{3.14}$$

This generalization has been related to the octonions because the solutions of (3.13) will be not only vacuum solutions of the Einstein equation without cosmological constant, but Ricci-flat, which is one of the main features of the G_2 manifolds. This follows directly from (3.13), the antisymmetry of (3.14) and the Bianchi identity $R_{d[ebc]} = 0$ as

$$R_{ab} = R_{acbc} = \frac{1}{2}c_{acde}R_{debc} = \frac{1}{2}c_{acde}R_{d[ebc]} = 0\ . \tag{3.15}$$

Indeed, it will be shown in the next section that seven dimensional self-dual manifolds have restricted holonomy G_2. For such case the Weyl tensor C_{abcd} defined by

$$C_{abcd} = R_{abcd} + \frac{R}{(n-1)(n-2)}(g_{ac}g_{bd} - g_{ad}g_{bc}) - \frac{1}{(n-2)}(g_{ac}R_{bd} - g_{ad}R_{bc} - g_{bc}R_{ad} + g_{bd}R_{ac}) \tag{3.16}$$

will be equal to the Riemann tensor. The first one is traceless, in consequence Ricci-flat manifolds have traceless curvature tensor.

Many authors presents the seven dimensional self-duality as a property of the spin connection, namely

$$\omega_{ab} = \frac{1}{2}c_{abcd}\omega_{cd}, \tag{3.17}$$

or, equivalently,

$$c_{abc}\omega_{bc} = 0. \tag{3.18}$$

In [32] a clear proof that the definition (3.17) implies (3.13) was given. It is based in the following identity for the octonion constants

$$c_{abcp}c_{defp} = -3c_{ab[de}\delta_{e]b} - 2c_{def[a}\delta_{b]c} + 6\delta_a^{[d}\delta_b^e\delta_c^{f]} - 2c_{def[a}\delta_{b]c}. \tag{3.19}$$

If (3.17) holds it is clear that $d\omega^a_b$ will be self-dual. The self-duality of $\omega^a_c \wedge \omega^c_b$ follows using that

$$\frac{1}{2}c_{abcd}\omega^c_e \wedge \omega^e_d = \frac{1}{4}c_{abcd}c_{edfg}\omega^c_e \wedge \omega^f_g$$

$$= -\frac{1}{2}c_{abfe}\omega^f_c \wedge \omega^c_e + \frac{1}{4}c_{aefg}\omega^b_e \wedge \omega^f_g - \frac{1}{4}c_{befg}\omega^a_e \wedge \omega^f_g + \omega^a_c \wedge \omega^c_b \tag{3.20}$$

$$= -\frac{1}{2} c_{abcd} \omega^c_e \wedge \omega^e_d + 2\omega^a_c \wedge \omega^c_b \, .$$

The identity (3.19) has been used here. From (3.20) is seen that

$$\frac{1}{2} c_{abcd} \omega^c_e \wedge \omega^e_d = \omega^a_c \wedge \omega^c_b$$

which implies the self-duality of R. The definition (3.17) implies (3.13), but the converse is not necessarily true.

4. The holonomy group of the metric obtained.

The purpose of this section is to show that seven dimensional self-dual manifolds has restricted holonomy G_2. Holonomy is the process of assigning to each closed curve of a manifold the linear transformation that measures the rotation resulting when a spinor or vector field is parallel transported around the given curve. The set of holonomy matrices constitutes a group, the holonomy group of the manifold. If it is considered only those curves which are contractible to a point it is the restricted holonomy. In simply connected manifolds both groups coincides.

From the Berger list [1] it follows that if a seven dimensional manifold admits only one covariantly constant spinor it will have G_2 restricted holonomy. It will be shown now that the seven dimensional self-dual manifolds admits exactly one. The covariant derivative of a spinor η is

$$D_i \eta = (\partial_i - \frac{1}{4} \omega_{iab} \gamma^{ab}) \eta \, . \tag{4.21}$$

Choosing its components as

$$\eta_\alpha = \delta_{\alpha 8} \tag{4.22}$$

it is obtained

$$D_i \eta = -\frac{1}{4} \omega_{iab} \gamma^{ab} \eta.$$

One of the possible representations of the $SO(7)$ gamma matrices is the antisymmetric and imaginary given by

$$\gamma^a_{\alpha\beta} = i(c'_{a\alpha\beta} + \delta_{a\alpha}\delta_{8\beta} + \delta_{a\beta}\delta_{8\alpha}),$$

where the constants $c'_{a\alpha\beta}$ are zero if α or β are equal to 8, and the octonion constants in other case. In this representation $(\gamma^{ab}\eta)_\alpha = c_{abc}$. Using (3.18) it follows that

$$D_i \eta = -\frac{1}{4} \omega_{iab} \gamma^{ab} \eta = -\frac{1}{4} \omega_{iab} c_{abc} = 0.$$

This result must be independent of the representation, which implies that the seven dimensional self-dual manifolds admits a covariantly constant spinor. A proof that this is the only spinor that can be defined will not be given here, the reference [4] can be consulted for the interested reader.

The change of an arbitrary field Ψ under infinitesimal parallel transport is

$$\delta\Psi = G_{ab} \delta A^{ab} \Psi \, , \tag{4.23}$$

where δA^{ab} is an infinitesimal area element spanned by the closed curve taking into consideration. $G_{ab} = R_{abcd}\Gamma^{cd}$ generate the infinitesimal holonomy group, being Γ^{cd} the generators of $SO(m)$ in the representation of the field Ψ. The restricted holonomy can be larger.

It has been mentioned in the introduction that for G_2 manifolds it is possible to construct a G_2 invariant closed and co-closed three form. For this reason it is needed to check that this holds for seven dimensional manifolds with self-dual spin connection. The most natural candidates to consider are the G_2 equivariant 3-form Φ (2.11) and its dual $*\Phi$

$$\Phi = \frac{1}{3!}c_{abc}e^a \wedge e^b \wedge e^c \ , \quad *\Phi = \frac{1}{4!}c_{abcd}e^a \wedge e^b \wedge e^c \wedge e^d. \tag{4.24}$$

Taking into account the identity

$$c_{abp}c_{pcde} = 3c_{a[cd}\delta_{e]b} - 2c_{b[cd}\delta_{e]a}$$

it is obtained

$$d\Phi = -\frac{1}{3!2}c_{abc}e^a \wedge e^b \wedge \omega^{cd} \wedge e^d = -\frac{1}{3!}c_{abc}c_{cdef}e^a \wedge e^b \wedge \omega^{ef} \wedge e^d$$

$$= \frac{1}{3!}c_{ade}e^a \wedge e^d \wedge \omega_{eb} \wedge e^b = -2d\Phi \ .$$

From here follows

$$d\Phi = 0 \ . \tag{4.25}$$

So, Φ is a closed form. Similarly, using (3.19) follows that

$$d * \Phi = -\frac{1}{4!6}c_{abcd}\omega_{ae} \wedge e^e \wedge e^b \wedge e^c \wedge e^d = -\frac{1}{4!12}c_{aefg}c_{abcd}\omega^{fg} \wedge e^e \wedge e^b \wedge e^c \wedge e^d$$

$$= \frac{1}{4!3}c_{fbcd}\omega^{fe} \wedge e^e \wedge e^b \wedge e^c \wedge e^d = -2d * \Phi \ ,$$

which implies the closure of $*\Phi$.

5. Construction of G_2 metrics with torus quaternionic manifolds as a base.

5.1 The Bryant-Salamon construction.

A construction of G_2 manifolds with four dimensional quaternionic Kahler manifold as a base space has been given in [33] and reconsidered in [34]. A quaternionic Kahler space is a Riemannian space of real dimension $4N$ endowed with a metric

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

and a set of three complex structures $J^{i\beta}_\alpha$ satisfying the quaternionic algebra

$$J^{i\beta}_\alpha J^{j\gamma}_\beta = -\delta_{ij}\delta^\gamma_\alpha + \epsilon_{ijk}J^{k\gamma}_\alpha. \tag{5.26}$$

The metric is quaternionic hermitian:

$$g(J^i X, J^i Y) = g(X, Y),$$

from where follows that $J^i_{\beta\alpha} = -J^i_{\alpha\beta}$. The holonomy group $H \subset Sp(n) \times Sp(1)$, and if the manifold has scalar curvature equal to zero it will be called hyperkahler. From the complex structures J^i is possible to construct the hyperkahler triplet of 2-forms given by

$$\Omega^i = \Omega^i_{\mu\nu} dx^\mu dx^\nu; \quad \Omega^i_{\mu\nu} = g_{\mu\omega}(J^i)^\omega_\nu.$$

Over quaternionic manifolds there are defined three local 1-forms

$$A^i = \omega^{mn} J^i_{mn} \tag{5.27}$$

where ω^{mn} represents the antiself-dual part of the spin connection. The hyperkahler form is covarianly closed with respect to the connection A^i ; this means that

$$\nabla_\alpha \Omega^i = d\Omega^i + \epsilon_{ijk} A^j \wedge \Omega^k = 0.$$

Defining the $SU(2)$ curvature

$$F^i = dA^i + \epsilon_{ijk} A^j \wedge A^k,$$

it follows that in the quaternionic kahler case

$$F^i = \kappa \Omega^i. \tag{5.28}$$

Here κ denotes the scalar curvature. Any quaternionic metric is an Einstein space with curvature κ and

$$R_{mn} = 3\kappa g_{mn}.$$

It is important for the purposes of the present work to remark that in four dimensions a quaternionic-Kahler metric is an Einstein metric with self-dual Weyl tensor. In the hyperkahler case $\kappa = 0$, so F^i is flat and the manifold is Ricci-flat.

Four dimensional quaternionic Kahler metrics can be used as base spaces to construct seven dimensional self-dual ones. It has been shown [33] that the seven dimensional metrics

$$ds^2 = \frac{1}{\sqrt{2\kappa|u|^2 + c}} (du^i + \epsilon^{ijk} A^j u^k)^2 + \sqrt{2\kappa|u|^2 + c}\, ds_4^2. \tag{5.29}$$

satisfies the self-duality condition (3.17) and in consequence the restricted holonomy will be G_2. The quaternionic base space has the metric ds_4^2; the total space is topologically an R^3 bundle with coordinates u_i. The curvature of the quaternionic manifold has been denoted by κ and c is an integration constant. In the hyperkahler limit $\kappa = 0$ it is possible to gauge away the connection A^i; the resulting manifold is the trivial product of R^3 by the hyperkahler manifold, which is non-compact and with holonomy G_2. Non-compact backgrounds are of interest in M-theory as well [34].

Expressing the R^3 part of (5.29) in polar coordinates it is obtained the following expression

$$ds^2 = \frac{dr^2}{(1 - \frac{4c}{r^4})} + \frac{r^2}{4\kappa}(1 - \frac{4c}{r^4}) g_{ab}(dx^a + \xi^a_i A^i)(dx^b + \xi^b_j A^j) + \frac{r^2}{2} ds_4^2,$$

where g_{ab} and ξ are the metric and the killing vectors of S^2 respectively. In this coordinate system is more clear that the metric is asymptotically a cone; in the limit $r \to \infty$ it is seen that

$$ds^2 \sim dr^2 + r^2 d\Omega$$

where the part related with Ω is independent of r. In other words, for large r (5.29) is a cone over a six dimensional space constructed as an S^2 bundle over a quaternionic kahler space with the metric ds_4^2. This six dimensional manifold has holonomy $SU(3)$ [6].

5.2 Toric quaternionic geometry in four dimensions.

It turns out from the previous discussion that quaternionic-kahler and hyperkahler manifolds are base spaces to construct G_2 holonomy metrics. The aim of this work is to investigate the case in which there are two commuting isometries. The hyperkahler case was discussed in [37] and is related with the monopole solution appearing in the context of the $(2+1)$ Einstein gravity [35]. The main result needed here is:

"The four dimensional euclidean metric

$$ds^2 = V_\eta(d\rho^2 + d\eta^2 + \rho^2 d\phi^2) + V_\eta^{-1}(d\psi + \rho V_\rho d\phi)^2 \tag{5.30}$$

will be hyperkahler if and only if the function V is an AHF, i.e, it satisfies the monopole equation

$$V_{\eta\eta} + \rho^{-1}(\rho V_\rho)_\rho = 0. \tag{5.31}$$

The metric (5.30) has an $U(1)\times U(1)$ isometry because the two killing spinors $\frac{\partial}{\partial\phi}$ and $\frac{\partial}{\partial\psi}$ commutes."

By another side, a method to construct non Ricci-flat four dimensional Einstein metrics with self-dual Weyl tensor and two commuting killing vectors has been investigated by Calderbank and Pedersen in [38]. This case is of interest as well, because such metrics will be quaternionic-kahler. Their statement is:

"Let $F(\rho,\eta)$ be a solution of the equation

$$F_{\rho\rho} + F_{\eta\eta} = \frac{3F}{4\rho^2}. \tag{5.32}$$

on some open subset of the half-space $\rho > 0$. Any Einstein-metric with self-dual Weyl tensor and nonzero scalar curvature possessing two linearly independent commuting Killing fields has locally the form

$$\begin{aligned} g = {} & \frac{F^2 - 4\rho^2(F_\rho^2 + F_\eta^2)}{4F^2}\frac{d\rho^2 + d\eta^2}{\rho^2} \\ & + \frac{[(F - 2\alpha F_\rho)\alpha - 2\rho F_\eta \beta]^2 + [-2\rho F_\eta\alpha + (F + 2\rho F_\rho)\beta]^2}{F^2[F^2 - 4\rho^2(F_\rho^2 + F_\eta^2)]}. \end{aligned} \tag{5.33}$$

where $\alpha = \sqrt{\rho}d\phi$ and $\beta = (d\psi + \eta d\phi)/\sqrt{\rho}$. On the open set defined by $F^2 > 4\rho^2(F_\rho^2 + F_\eta^2)$ g has positive scalar curvature, whereas $F^2 < 4\rho^2(F_\rho^2 + F_\eta^2)$ -g is self-dual with negative scalar curvature."

It follows that quaternionic metrics with torus symmetry has positive signature. The three 1-forms (5.27) have a remarkable simple expression in terms of F,

$$A^1 = \frac{1}{F}[-\rho F_\eta\frac{d\rho}{\rho} + (\frac{1}{2}F + \rho F_\rho)\frac{d\eta}{\rho}], \quad A^2 = \frac{\alpha}{F}, \quad A^3 = \frac{\beta}{F}. \tag{5.34}$$

and the relation 5.28 holds with $\kappa = 1$.

Both statements are associated to solve two linear equations of second order, namely (5.31) and (5.32). This equations are related by a Backglund transformation, which implies any solution of one of them allow us to construct a solution of the other one. This fact can be exploited to find examples of quaternionic metrics.

5.3 The Backglund transformation.

To prove that (5.31) and (5.32) are Backglund transformed it is needed to introduce the Joyce system of equations [38]

$$(S_0)_\rho + (S_1)_\eta = S_0/\rho, \quad (S_0)_\eta - (S_1)_\rho = 0 \tag{5.35}$$

where S_0 and S_1 are unknown functions. Selecting $S_0 = H_\rho$ and $S_1 = H_\eta$ the second equation will be trivial and the first became $H_{\rho\rho} + H_{\eta\eta} = H_\rho/\rho$ (G is usually called a Tod coordinate). Now, taking $H = \rho^{1/2}F$ it follows that F satisfies (5.32). Conversely, selecting $S_0 = -\rho V_\eta$ and $S_1 = \rho V_\rho$ the first equation is trivial and the second one is $V_{\eta\eta} + \rho^{-1}(\rho V_\rho)_\rho = 0$, in other words V is an AHF.

It is seen that a solution V of (5.31) allows construct a solution F of (5.32) integrating the system

$$-\rho V_\eta = H_\rho; \quad \rho V_\rho = H_\eta \tag{5.36}$$

and defining $F = H/\rho^{1/2}$. Conversely, a solution F of (5.32) allows to construct a solution V of (5.31) defining $H = \rho^{1/2}F$ and integrating (5.36). By construction the equation for F is the integrability condition for V and viceversa; the relation between them is an example of a Backglund transformation. This Backglund mapping gives a method to construct quaternionic-kahler metrics starting with an hyperkahler example and viceversa, in both cases there is a torus symmetry.

Certain properties of the AHF are known and this can be used to find new solutions of (5.32). It has been shown [35] that any solution V of the equation

$$V_{\eta\eta} - \rho^{-1}(\rho V_\rho)_\rho = 0$$

can be expressed in integral form as

$$V(\eta, \rho) = \frac{1}{2\pi}\int_0^{2\pi} G(\rho sen(\theta) + \eta)d\theta. \tag{5.37}$$

Here $G(x)$ denotes an arbitrary function of one variable. The function $V(i\eta, \rho)$ will be a solution of (5.31), and $W(\eta, \rho) = V(i\eta, \rho) + V(-i\eta, \rho)$ will be a real AHF. In the following subsection this facts will be applied to construct quaternionic spaces starting with hyperkahler ones and solving the Backglund equations (5.36).

5.4 Examples of quaternionic spaces with torus symmetry.

The following are quaternionic metrics constructed with the method explained in the previous subsection:

A. The trivial four dimensional toric metric

$$ds^2 = d\rho^2 + d\eta^2 + \rho^2 d\phi^2 + d\psi^2$$

corresponds to the monopole

$$V = \eta$$

(This AHF holds using $G(x) = xLog(x)$ in the integral expression (5.37)). The equations (5.36) are in this case

$$-\rho = H_\rho; \quad H_\eta = 0,$$

and the eigenfunction F is given by

$$F = \frac{\rho^{3/2}}{2}.$$

The insertion of the last eigenfunction in (5.33) gives the following metric:

$$g = -\frac{2d\rho^2 + 2d\eta^2}{\rho^2} - \frac{\rho^2(1+3\rho) + \eta^2(9+7\rho)}{2\rho^5}d\phi^2 - \frac{8}{\rho^4}d\psi^2 - \frac{16\eta}{\rho^4}d\phi d\psi. \tag{5.38}$$

The inequality $F^2 < 4\rho^2(F_\rho^2 + F_\eta^2)$ holds for $\rho > 0$. Invoking the alderbank-Pedersen theorem, we see that for $\rho > 0$ the quaternionic kahler metric is -g. In this case $\kappa = -1$ and the three 1-forms A^i are

$$A^1 = \frac{2d\eta}{\rho}; \quad A^2 = -\frac{2d\phi}{\rho}; \quad A^3 = \frac{2\eta}{\rho^2}d\phi + \frac{2}{\rho^2}d\psi.$$

The metric constructed in this example is defined for all the positive values of ρ.

B. With the function $G(x) = Log(x)x^3$ it is found the monopole

$$V = 3\eta\rho^2 - 2\eta^3$$

and the hyperkahler metric

$$ds^2 = (3\rho^2 - 6\eta^2)(d\rho^2 + d\eta^2 + \rho^2 d\phi^2) + \frac{1}{3\rho^2 - 6\eta^2}(d\psi + 6\eta\rho^2 d\phi)^2.$$

The Backglund transformed F results

$$F = \frac{3}{4}\rho^{3/2}(4\eta^2 - \rho^2)$$

and the corresponding metric is

$$g = g_{\rho\rho}(d\rho^2 + d\eta^2) + g_{\phi\phi}d\phi^2 + g_{\psi\psi}d\psi^2 + 2g_{\phi\psi}d\phi d\psi, \tag{5.39}$$

where the components of the metric tensor are

$$g_{\rho\rho} = -\frac{4(8\eta^4 + 6\eta^2\rho^2 + 3\rho^4)}{(\rho^3 - 4\eta^2\rho)^2}$$

$$g_{\phi\phi} = -\frac{8\eta^4\rho^2(19+5\rho) + 16\eta^6(9+7\rho) + \rho^6(1+35\rho) + 3\eta^2\rho^4(35+61\rho)}{9\rho^5(\rho^2 - 4\eta^2)^2(8\eta^4 + 6\eta^2\rho^2 + 3\rho^4)}$$

$$g_{\psi\psi} = -\frac{64(4\eta^4 + \rho^4)}{9\rho^4(\rho^2 - 4\eta^2)^2(8\eta^4 + 6\eta^2\rho^2 + 3\rho^4)}$$

$$g_{\psi\phi} = -\frac{32(8\eta^5 - \rho^2\eta^3 + 3\eta\rho^4)}{9\rho^4(\rho^2 - 4\eta^2)^2(8\eta^4 + 6\eta^2\rho^2 + 3\rho^4)}.$$

As in the example A, $F^2 < 4\rho^2(F_\rho^2 + F_\eta^2)$ for $\rho > 0$, -g is a quaternionic kahler metric and $\kappa = -1$. The three forms A^i are given by

$$A^1 = \frac{8\eta}{\rho^2 - 4\eta^2}d\rho + \frac{4(\rho^2 - 2\eta^2)}{\rho(\rho^2 - 4\eta^2)}d\eta$$

$$A^2 = \frac{4}{3\rho(\rho^2-4\eta^2)}d\phi; \quad A^3 = -\frac{4\eta}{3\rho(\rho^2-4\eta^2)}d\phi - \frac{4}{3\rho(\rho^2-4\eta^2)}d\psi.$$

The metric (5.39) is singular at $\rho \to 0$ and at the lines $2|\rho| = |\eta|$.

C. The powers $G(x) = x^n$ and $G(x) = Log(x)x^{2n+1}$ can be integrated out giving polynomial solutions of higher degree. For instance $G(x) = Log(x)x^5$ gives

$$V = 3\eta b\rho^2 - 2\eta^3; \quad F = \frac{1}{\eta}(8\eta^4 + 40\eta^2\rho^2 + 15\rho^4).$$

The even powers $G(x) = x^{2n}Log(x)$ can be integrated too, but the expressions of the metrics are more complicated by the appearance of logarithm terms. For example, with $G(x) = Log(x)x^2$ it is obtained

$$V = 6\eta^2 - \rho^2 - 6\eta^2\sqrt{1+\frac{\rho^2}{\eta^2}} + 2(\rho^2 - 2\eta^2)Log(\frac{2}{\eta+\sqrt{\eta^2+\rho^2}})$$

$$F = \frac{4}{3}\eta^3 - 4\eta\rho^2 - \frac{4}{3}\eta^2(\eta^2+\rho^2) + \frac{8}{3}\rho^2(\eta^2+\rho^2) + 4\eta\rho^2 Log(\frac{2}{\eta+\sqrt{\eta^2+\rho^2)}}).$$

The reader can check that the expression for the quaternionic metric is very large and difficult to simplify.

D. The function $G(x) = e^x$ gives

$$V = e^{\pm i\eta}I_0(\rho) + c.c$$

where $I_n(\rho)$ denotes the modified Bessell function of the first kind, which are solutions of the equation

$$\rho^2 H''(\rho) + \rho H'(\rho) - (\rho^2 + n^2)H(\rho) = 0.$$

The hyperkahler space that corresponds to this monopole is:

$$ds^2 = \rho I_1(\rho)cos(\eta)(d\rho^2 + d\eta^2 + \rho^2 d\phi^2)$$

$$+\frac{1}{\rho I_1(\rho)cos(\eta)}\{d\psi + \rho[I_1(\rho) + \frac{\rho}{2}(I_0(\rho) + I_2(\rho))]sin(\eta)d\phi\}^2. \tag{5.40}$$

The Backglund transformed eigenfunction F is given by

$$F = \sqrt{\rho}I_1(\rho)e^{\pm i\eta} + c.c.$$

The metric

$$ds_{qk} = \frac{\Theta(\rho,\eta)}{4\rho I_1(\rho)^2}(d\rho^2 + d\eta^2) + \frac{[2\rho^{3/2}cos(\eta)I_1(\rho)\beta - \rho^{3/2}sin(\eta)(I_0(\rho)+I_2(\rho))\alpha]^2}{\Phi(\rho,\eta)}$$

$$+\frac{[2\rho^{3/2}cos(\eta)I_1(\rho)\alpha + \sqrt{\rho}sin(\eta)(\rho I_0(\rho) + 2I_1(\rho) + \rho I_2(\rho))\beta]^2}{\Phi(\rho,\eta)}, \tag{5.41}$$

where it has been defined

$$\Theta(\rho,\eta) = \rho I_0(\rho)^2 + 2I_1(\rho)I_2(\rho) + \rho I_2(\rho)^2 + 4\rho I_1(\rho)^2 ctg(\eta)^2 + 2I_0(\rho)(I_1(\rho) + \rho I_2(\rho))$$

and

$$\Phi(\rho,\eta)=\rho^2 I_1(\rho)^2 sin(\eta)^2[-4\rho^2 I_1(\rho)^2 cos(\eta)^2+I_1(\rho)^2 sin(\eta)^2-(\rho I_0(\rho)+I_1(\rho)+\rho I_2(\rho))^2 sin(\eta)^2],$$

will be quaternionic for the regions of the plane (ρ,η) in which $F^2<4\rho^2(F_\rho^2+F_\eta^2)$. In those regions $k=-1$. The three one forms A^i are

$$A^1=\frac{1}{2}[1+tg(\eta)+\frac{\rho}{I_1(\rho)}tg(\eta)(I_0(\rho)+I_2(\rho))]\frac{d\eta}{\rho}-tg(\eta)d\rho$$

$$A^2=\frac{\alpha}{\sqrt{\rho}cos(\eta)I_1(\rho)};\quad A^3=\frac{\beta}{\sqrt{\rho}cos(\eta)I_1(\rho)}.$$

The radial component of the metric (5.41) shows that some of the singularities are the zeros of $I_1(\rho)$.

E. In the previous examples it have been constructed quaternionic kahler metrics starting with an hyperkahler one. In the present one it will be constructed hyperkahler metrics corresponding to the m-pole solutions investigated in [40] and [38]. In the first reference of [38] it has been described the moduli space corresponding to the 3-pole solutions and has been shown that they encode some well known examples appearing in the physics, like the Bianchi type spaces. Moreover, it has been analyzed by Angeluova and Lazaroiu the dynamics of the M-theory on toric G_2 cones constructed with m-pole spaces as base manifolds [43]. For those reasons it may be useful for the reader a brief review of the main features of such manifolds. It will be shown that the 3-pole hyperkahler metrics coincides with those discussed in [37]. The following exposition follows closely those given in the references [38].

The basic eigenfunctions F of (5.32) are

$$F(\rho,\eta,y)=\frac{\sqrt{(\rho)^2+(\eta-y)^2}}{\sqrt{\rho}} \tag{5.42}$$

where the parameter y takes arbitrary real values. Using the Backglund transformation it is found the basic monopole

$$V(\eta,\rho,y)=-Log[\eta-y+\sqrt{\rho^2+(\eta-y)^2}]. \tag{5.43}$$

Being the equations for F and V linear, for any set of real numbers w_i the functions

$$F=\sum_{j=0}^{k+1}w_i F(\rho,\eta,y_j). \tag{5.44}$$

$$V=\sum_{j=0}^{k+1}w_i V(\rho,\eta,y_j) \tag{5.45}$$

will be solutions too. For this reason the 2-pole functions given by

$$F_1=\frac{1+\sqrt{\rho^2+\eta^2}}{\sqrt{\rho}};\quad F_2=\frac{\sqrt{(\rho)^2+(\eta+1)^2}}{\sqrt{\rho}}-\frac{\sqrt{(\rho)^2+(\eta-1)^2}}{\sqrt{\rho}},$$

are eigenfunctions of the hyperbolic laplacian. The first one gives rise to the spherical metric, while the second one gives rise to the hyperbolic metric

$$ds^2 = (1 - r_1^2 - r_2^2)^{-2}(dr_1^2 + dr_2^2 + r_1^2 d\theta_1^2 + r_2^2 d\theta_2^2).$$

The relation between the coordinates (r_1, r_2) and (ρ, η) can be extracted from the relation

$$(r_1 + ir_2)^2 = \frac{\eta - 1 + i\rho}{\eta + 1 + i\rho}.$$

The hyperkahler metrics corresponding to both cases are

$$ds^2 = -\frac{1}{\sqrt{\rho^2 + \eta^2}}(d\rho^2 + d\eta^2 + \rho^2 d\phi^2) - \sqrt{\rho^2 + \eta^2}(d\psi + \frac{\eta}{\sqrt{\rho^2 + \eta^2}}d\phi)^2, \tag{5.46}$$

and

$$ds^2 = \frac{\sqrt{\rho^2 + (\eta-1)^2} - \sqrt{\rho^2 + (\eta+1)^2}}{\sqrt{\rho^2 + (\eta+1)^2}\sqrt{\rho^2 + (\eta-1)^2}}(d\rho^2 + d\eta^2 + \rho^2 d\phi^2)$$

$$+\frac{\sqrt{\rho^2 + (\eta+1)^2}\sqrt{\rho^2 + (\eta-1)^2}}{\sqrt{\rho^2 + (\eta-1)^2} - \sqrt{\rho^2 + (\eta+1)^2}}[d\psi + (\frac{\eta+1}{\sqrt{\rho^2 + (\eta+1)^2}} - \frac{\eta-1}{\sqrt{\rho^2 + (\eta-1)^2}})d\phi]^2. \tag{5.47}$$

The general "3-pole" solutions are

$$F = \frac{1}{\sqrt{\rho}} + \frac{b + c/m}{2}\frac{\sqrt{\rho^2 + (\eta+m)^2}}{\sqrt{\rho}} + \frac{b - c/m}{2}\frac{\sqrt{\rho^2 + (\eta-m)^2}}{\sqrt{\rho}}.$$

By definition $-m^2 = \pm 1$, which means that m can be imaginary or real. The corresponding solutions are denominated type I and type II respectively. It is convenient to introduce the Eguchi-Hanson like coordinate system defined by

$$\rho = \sqrt{R^2 \pm 1}cos(\theta), \quad \eta = Rsen(\theta),$$

where θ takes values in the interval $(-\pi/2, \pi/2)$. In this coordinates

$$\sqrt{\rho}F = 1 + bR + csin(\theta), \tag{5.48}$$

$$\rho^{-1}[\frac{1}{4}F^2 - \rho^2(F_\rho^2 + F_\eta^2)] = \frac{b(R \mp b) + c(sin(\theta) + c)}{R^2 \pm sin^2(\theta)}. \tag{5.49}$$

and the family of self-dual metrics corresponding to the 3-pole are expressed as

$$ds^2 = \frac{b^2 - c^2 + (bR - cS)}{(1 + bR + cS)^2}(\frac{dR^2}{R^2 - 1} + \frac{dS^2}{1 - S^2})$$

$$\frac{1}{(1 + bR + cS)^2(b^2 - c^2 + (bR - cS))(R^2 - S^2)}$$

$$*((R^2 - 1)(1 - S^2)((bR - cS)d\varphi + (cR - bS)d\psi)^2$$

$$+((b(R^2 - 1)S + c(1 - S^2)R)d\varphi + (c(R^2 - 1)S + b(1 - S^2)R + (R^2 - S^2)d\psi)^2) \tag{5.50}$$

It has been denoted $S = sin(\theta)$ here. The expression (5.50) includes some well known metrics. Let us focus in the type I case. The formulas (5.48) and (5.49) allows to determine the domain of definition of the metric (5.33). When b is nonzero for a given value of θ, $F = 0$ if $R = -(1 + csin(\theta))/b$ and $(\frac{1}{4}F^2 - \rho^2(F_\rho^2 + F_\eta^2)) = 0$ if $R = (b^2 + c^2 + csin(\theta))/b$. The case $c = 0$ correspond to a bi-axial Bianchi IX metric [41]. The domains of definition are $(-\infty, R_\infty), (R_\infty, R_\pm)$, and $(R_\pm, \infty)$. In the first two cases the curvature is negative, and in the last one positive, and in the two last cases there is an unremovable singularity at $R = R_\pm$. In the case $b = 0$ for $c > 1$ and $c < 1$ the metric will be of Bianchi VIII type [39]. The case $c = 1$ corresponds to the Bergmann metric on CH^2.

For the type II case, the range of R is $(1, \infty)$ but the moduli space is more complex that in the type I case. For the lines $b = \pm c$ it is obtained the hyperbolic metric if $b < 0$ and the spherical metric if $b > 0$. If $(b, c) = (1, 0)$ it is obtained the Fubbini-Study metric on CP^2 whereas the points $(0, 1), (-1, 0)$ and $(0, -1)$ yield again the Bergmann metric on CH^2. Along the lines joining $(1, 0)$ with others we have bi-axial Bianchi metric IX, while along the lines between $(0, 1), (-1, 0)$ and $(0, -1)$ the metric is Bianchi VIII. A more complete description is given in [38].

The triplet of one forms corresponding to this family of metrics is

$$A^1 = A^1_+ + A^1_-,$$

$$A^2 = \frac{\sqrt{(R^2 \pm 1)(1 - S^2)}}{(1 + bR + cS)} d\phi \quad A^3 = \frac{d\psi + \eta d\phi}{(1 + bR + cS)},$$

where it has been defined

$$A^1_\pm = \frac{1}{2(1 + bR + cS)}[(H_\pm R + \frac{2SK_\pm}{\sqrt{1 - S^2}})\frac{dR}{\sqrt{R^2 \pm 1}} + (\frac{2K_\pm}{\sqrt{(R^2 \pm 1)(1 - S^2)}} - \sqrt{R^2 \pm 1} SH_\pm)dS],$$

$$K_\pm = \frac{2(bm \pm c)(R^2 \pm 1)(1 - S^2)}{m\sqrt{(R^2 \pm 1)(1 - S^2) + (RS \pm m)^2}} + \frac{(c \pm bm)\sqrt{(R^2 \pm 1)(1 - S^2) + (m \pm RS)^2}}{2m},$$

$$H_\pm = \frac{(b \pm c/m)(RS \pm m)}{\sqrt{(R^2 \pm 1)(1 - S^2) + (m \pm RS)^2}}.$$

(The reader should care that the $\pm$ in $(R^2 \pm 1)$ depends only on the metric in consideration, it is $+$ for type I and $-$ for type II.) The Backglund transformed function V reads

$$V = Log(\rho) + \frac{1}{2}(b + c/m)Log[\frac{\eta - m + \sqrt{(\eta - m)^2 + \rho^2}}{\rho}] + \frac{1}{2}(b - c/m)Log[\frac{\eta + m + \sqrt{(\eta + m)^2 + \rho^2}}{\rho}].$$

For the type I case this is the potential for an axially symmetric circle of charge, while the type II case corresponds to two point sources on the axis of symmetry. The hyperkahler metrics obtained are encoded in the following expression

$$ds^2 = \frac{bR + c\sqrt{1 - S^2}}{R^2 \pm (1 - S^2)}(d\rho^2 + d\eta^2 + \rho^2 d\phi^2)$$

$$+ \frac{R^2 \pm (1 - S^2)}{bR + c\sqrt{1 - S^2}}[d\psi + \frac{R^2 \pm (1 - S^2) - b(R^2 \pm 1)\sqrt{1 - S^2} + cRS^2}{R^2 \pm (1 - S^2)} d\phi]^2. \tag{5.51}$$

This manifolds have been investigated recently in [37] and it has been shown that the quotient of (5.51) with $\frac{\partial}{\partial\phi}$ gives the Eguchi-Hanson type Einstein-Weyl metrics in D=3.

The continuum limit of the expressions (5.44) and (5.45) are

$$F(\rho,\eta)=\int w(y)F(\rho,\eta,y)dy. \tag{5.52}$$

$$V(\rho,\eta)=\int w(y)V(\rho,\eta,y)dy. \tag{5.53}$$

where $w(y)$ is a distribution with compact support in R. A choice of $w(y)$ for which at least one of the integrals (5.52) and (5.53) converges gives rise to an smooth solution. For instance for $w(y)=y/(y^2+1)^2$ it is obtained the following non-trivial monopole

$$V(\rho,\eta)=\frac{cos(\frac{1}{2}Arg(1-2i\eta-\eta^2-\rho^2))Log(\frac{|1-i\eta-\sqrt{(1-i\eta)^2+\rho^2}|}{|1+i\eta-\sqrt{(1-i\eta)^2+\rho^2}|})}{\sqrt{|(1-i\eta)^2+\rho^2|}}$$

$$+\frac{sin(\frac{1}{2}Arg(1-2i\eta-\eta^2-\rho^2))Arg(\frac{1-i\eta-\sqrt{(1-i\eta)^2+\rho^2}}{1+i\eta-\sqrt{(1-i\eta)^2+\rho^2}})}{\sqrt{|(1-i\eta)^2+\rho^2|}},$$

and from (5.30) follows an hyperkahler metric. But (5.52) is divergent for this distribution.

To conclude this subsection it should be mentioned that higher m-pole solutions have been considered in [43], and that quaternionic spaces with torus symmetry have been investigated recently in [42] using the harmonic space formalism.

5.5 G_2 holonomy metrics with torus symmetry and supergravity backgrounds.

In this subsection will be constructed the G_2 holonomy metrics corresponding to the examples A and B. After extend them to a vacuum configuration of the eleven dimensional supergravity it will be obtained type IIA backgrounds by reduction along one of the isometries.

In [43] it has been found the explicit form of (5.29) when the base space (and, in consequence, the total one) has torus symmetry. The expression is

$$ds^2=\frac{dr^2}{h(r)}+\frac{r^2}{2}[U_{\phi\phi}d\phi^2+U_{\phi\psi}d\phi d\psi+U_{\psi\psi}d\psi^2+Q_\phi d\phi+Q_\psi d\psi+g_{\rho\rho}(d\eta^2+d\rho^2)+H]. \tag{5.54}$$

where it has been defined

$$h(r)=1-4c/r^4$$

$$U_{11}=g_{\phi\phi}+h(r)\frac{u_1^2(\rho^2+\eta^2)+(u_2\eta+u_3\rho)^2}{2\rho F^2},$$

$$U_{22}=g_{\psi\psi}+h(r)\frac{u_1^2+u_2^2}{2\rho F^2},$$

$$U_{12}=U_{21}=g_{\phi\psi}+h(r)\frac{(u_1^2+u_2^2)\eta+u_2u_3\rho}{2\rho F^2},$$

$$Q_\phi = h(r)\frac{1}{\sqrt{\rho}F}[u_1(\eta du_2 + \rho du_3) - (u_2\eta + u_3\rho)du_1 - u_1(u_3\eta - u_2\rho)A^1],$$

$$Q_\psi = h(r)[\frac{u_1 du_2 - u_2 du_1 - u_1 u_3 A^1}{\sqrt{\rho}F}],$$

$$H = h(r)[|d\overrightarrow{u}|^2 + (u_1^2 + u_2^2)(A^1)^2 - 2A^1(u_3 du_2 - u_2 du_3)].$$

The second rank tensor g_{ab} is the metric of the base manifold. The product metric of (5.54) with M^4

$$ds_{11}^2 = ds_M^2 + \frac{dr^2}{h(r)} + \frac{r^2}{2}[U_{\phi\phi}d\phi^2 + U_{\phi\psi}d\phi d\psi + U_{\psi\psi}d\psi^2 + Q_\phi d\phi + Q_\psi d\psi + g_{\rho\rho}(d\eta^2 + d\rho^2) + H]. \quad (5.55)$$

is a vacuum configuration of the eleven dimensional supergravity [13]. With the help of the quantities

$$\alpha_1 = \frac{U_{22}Q_\phi - U_{12}Q_\psi}{detU}, \quad \alpha_2 = \frac{U_{11}Q_\psi - U_{12}Q_\phi}{detU},$$

$$h = H + U_{11}\alpha_1^2 + 2U_{12}\alpha_1\alpha_2 + U_{22}\alpha_2^2,$$

$$\phi_1 = \phi, \quad \phi_2 = \psi,$$

the metric (5.55) is expressed in more simple manner as

$$ds_{11}^2 = ds_{M^4}^2 + \frac{dr^2}{h(r)} + \frac{r^2}{2}[U_{ij}(d\phi_i + \alpha_i)(d\phi_j + \alpha_j) + h].$$

The last expression takes the usual form of the Kaluza-Klein anzatz

$$ds_{11}^2 = e^{-\frac{2}{3}\varphi_D}G_{\mu\nu}dx^\nu dx^\mu + e^{\frac{4}{3}\varphi_D}(d\phi + dx^\mu C_\mu(x))^2. \quad (5.56)$$

with the dilaton field and the RR 1-form defined by

$$\varphi_D = \frac{3}{4}Log(\frac{r^2 U_{11}}{2}),$$

$$C = \frac{U_{12}d\psi + Q_\phi}{U_{11}}.$$

The reduction of (5.56) along ϕ_1 gives the following IIA metric:

$$ds_A^2 = (\frac{r^2 U_{11}}{2})^{1/2}\{ds_{M^4}^2 + \frac{dr^2}{h(r)} + \frac{r^2}{2U_{11}}[detU d\psi^2 + 2(U_{11}Q_\psi - U_{12}Q_\phi)d\psi - Q_\phi^2 + U_{11}H]\}. \quad (5.57)$$

The components of (5.57) are

$$g_{\psi\varphi} = (\frac{r^2 U_{11}}{2})^{1/2}\frac{r^2}{4U_{11}F\sqrt{\rho}}h(r)[U_{11}sin^2(\theta) - U_{12}(\frac{\rho}{2}sin(2\theta)sin(\varphi) + \eta sin^2(\theta))]$$

$$g_{\psi\theta} = (\frac{r^2 U_{11}}{2})^{1/2}\frac{r^2}{4U_{11}F}h(r)U_{12}\sqrt{\rho}cos(\varphi)$$

$$g_{\theta\theta} = (\frac{r^2 U_{11}}{2})^{1/2}[\frac{r^2 h(r)}{4} - \frac{r^2 h^2(r)}{8U_{11}\rho F^2}\rho^2 cos^2(\varphi)]$$

$$g_{\varphi\varphi}=(\frac{r^2U_{11}}{2})^{1/2}[\frac{r^2h(r)}{4}sin^2(\theta)-\frac{r^2h^2(r)}{8U_{11}\rho F^2}sin^2(\theta)(\eta sin(\theta)+\rho cos(\theta)sin(\varphi))^2]$$

$$g_{\varphi\theta}=(\frac{r^2U_{11}}{2})^{1/2}\frac{r^2h^2(r)}{4U_{11}F^2}sin(\theta)cos(\varphi)(\eta sin(\theta)+\rho cos(\theta)sin(\varphi)),$$

where it have been introduced the spherical coordinates θ, φ through the relations

$$u_1=sin(\theta)cos(\varphi),\quad u_2=sin(\theta)sin(\varphi),\quad u_3=cos(\theta).$$

The range of this coordinates is $\theta\in[0,\pi]$ and $\varphi\in[0,2\pi]$; the other components of the metric are identically zero.

The base metric (5.38) have a singularity at $\rho\to 0$. For this case it is obtained

$$U_{11}=\frac{\rho^2(1+3\rho)+\eta^2(9+7\rho)}{2\rho^5}+2h(r)[\frac{u_1^2(\rho^2+\eta^2)+(u_2\eta+u_3\rho)^2}{\rho^4}]\sim\frac{f(x^i)}{\rho^5},$$

$$U_{22}=\frac{8}{\rho^4}+2h(r)(\frac{u_1^2+u_2^2}{\rho^4})\sim\frac{f(x^i)}{\rho^4},$$

$$U_{12}=U_{21}=\frac{8\eta}{\rho^4}+h(r)[\frac{(u_1^2+u_2^2)\eta+u_2u_3\rho}{\rho^4}]\sim\frac{f(x^i)}{\rho^4},$$

$$Q_\phi=h(r)\frac{2}{\rho^2}[u_1(\eta du_2+\rho du_3)-(u_2\eta+u_3\rho)du_1$$

$$-u_1(u_3\eta-u_2\rho)\frac{2d\eta}{\rho}]\sim\frac{f(x^i,dx^i)}{\rho^2},$$

$$Q_\psi=h(r)\frac{1}{\rho^2}(u_1du_2-u_2du_1-u_1u_3\frac{2d\eta}{\rho})\sim\frac{f(x^i,dx^i)}{\rho^2},$$

$$H=h(r)[|d\overrightarrow{u}|^2+4(u_1^2+u_2^2)\frac{d\eta^2}{\rho^2}-4\frac{d\eta}{\rho}(u_3du_2-u_2du_3)]\sim\frac{f(x^i,dx^i)}{\rho^2}.$$

where x^i denotes all the coordinates except ρ and the behaviour for short distances was evaluated. The dilaton field is given explicitly as

$$\varphi_A=\frac{3}{4}Log\{\frac{r^2}{2}[\frac{\rho^2(1+3\rho)+\eta^2(9+7\rho)}{2\rho^5}]+r^2h(r)[\frac{u_1^2(\rho^2+\eta^2)+(u_2\eta+u_3\rho)^2}{\rho^4}]\}\sim Log(\frac{f(x^i)}{\rho^5}).$$

The expression for the RR one form is

$$C=\frac{2\rho\{8\eta+h(r)[(u_1^2+u_2^2)\eta+u_2u_3\rho]\}d\psi}{\rho^2(1+3\rho)+\eta^2(9+7\rho)+4\rho h(r)[u_1^2(\rho^2+\eta^2)+(u_2\eta+u_3\rho)^2]}$$

$$+\frac{4\rho^2h(r)[\rho u_1(\eta du_2+\rho du_3)-\rho(u_2\eta+u_3\rho)du_1-2u_1(u_3\eta-u_2\rho)d\eta]}{\rho^2(1+3\rho)+\eta^2(9+7\rho)+4\rho h(r)[u_1^2(\rho^2+\eta^2)+(u_2\eta+u_3\rho)^2]}\sim f(x^i)\rho.$$

The components of (5.57) diverges in this case for short ρ,

$$g_{\psi\varphi}\sim\frac{f(x^i)}{\rho^5},\quad g_{\psi\theta}\sim\frac{f(x^i)}{\rho^4},\quad g_{\theta\theta}\sim\frac{f(x^i)}{\rho^2}$$

$$g_{\varphi\varphi}\sim\frac{f(x^i)}{\rho^2},\quad g_{\varphi\theta}\sim\frac{f(x^i)}{\rho}.$$

The quaternionic space (5.39) is singular too in the limit $\rho \to 0$. Using it as a base space gives

$$U_{11}=\frac{8\eta^4\rho^2(19+5\rho)+16\eta^6(9+7\rho)+\rho^6(1+35\rho)+3\eta^2\rho^4(35+61\rho)}{9\rho^5(\rho^2-4\eta^2)^2(8\eta^4+6\eta^2\rho^2+3\rho^4)}$$

$$+8h(r)[\frac{u_1^2(\rho^2+\eta^2)+(u_2\eta+u_3\rho)^2}{9\rho^4(\rho^2-4\eta^2)^2}]\sim\frac{f(x^i)}{\rho^5},$$

$$U_{22}=\frac{64(4\eta^4+\rho^4)}{9\rho^4(\rho^2-4\eta^2)^2(8\eta^4+6\eta^2\rho^2+3\rho^4)}+8h(r)[\frac{u_1^2+u_2^2}{9\rho^4(\rho^2-4\eta^2)^2}]\sim\frac{f(x^i)}{\rho^4},$$

$$U_{12}=U_{21}=\frac{32(8\eta^5-4\eta^3\rho^2+3\eta\rho^4)}{9\rho^4(\rho^2-4\eta^2)^2(8\eta^4+6\eta^2\rho^2+3\rho^4)}+8h(r)[\frac{(u_1^2+u_2^2)\eta+u_2u_3\rho}{9\rho^4(\rho^2-4\eta^2)^2}]\sim\frac{f(x^i)}{\rho^4},$$

$$Q_\phi=4h(r)\frac{1}{3\rho^2(4\eta^2-\rho^2)}\{u_1(\eta du_2+\rho du_3)-(u_2\eta+u_3\rho)du_1$$

$$-u_1(u_3\eta-u_2\rho)[\frac{8\eta}{\rho^2-4\eta^2}d\rho+\frac{4(\rho^2-2\eta^2)}{\rho(\rho^2-4\eta^2)}d\eta]\}\sim\frac{f(x^i,dx^i)}{\rho^2},$$

$$Q_\psi=h(r)\frac{4}{3\rho^2(4\eta^2-\rho^2)}\{u_1du_2-u_2du_1-u_1u_3[\frac{8\eta}{\rho^2-4\eta^2}d\rho+\frac{4(\rho^2-2\eta^2)}{\rho(\rho^2-4\eta^2)}d\eta]\}\sim\frac{f(x^i,dx^i)}{\rho^2},$$

$$H=h(r)\{|d\overrightarrow{u}|^2+(u_1^2+u_2^2)[\frac{8\eta}{\rho^2-4\eta^2}d\rho+\frac{4(\rho^2-2\eta^2)}{\rho(\rho^2-4\eta^2)}d\eta]^2-2[\frac{8\eta}{\rho^2-4\eta^2}d\rho$$

$$+\frac{4(\rho^2-2\eta^2)}{\rho(\rho^2-4\eta^2)}d\eta](u_3du_2-u_2du_3)\}\sim\frac{f(x^i,dx^i)}{\rho}.$$

The dilaton field is expressed through the relation

$$e^{\frac{4}{3}\varphi_D}=\frac{r^2}{2}\{\frac{8\eta^4\rho^2(19+5\rho)+16\eta^6(9+7\rho)+\rho^6(1+35\rho)+3\eta^2\rho^4(35+61\rho)}{9\rho^5(\rho^2-4\eta^2)^2(8\eta^4+6\eta^2\rho^2+3\rho^4)}$$

$$+8h(r)[\frac{u_1^2(\rho^2+\eta^2)+(u_2\eta+u_3\rho)^2}{9\rho^4(\rho^2-4\eta^2)^2}]\},$$

from where follows that

$$\varphi_D\sim Log[\frac{f(x^i)}{\rho^5}],\qquad \rho\to 0.$$

The behaviour of the RR one-form at short distances results

$$C\sim f(x^i,dx^i)\rho.$$

and the components of the IIA metric diverges as

$$g_{\psi\varphi}\sim\frac{f(x^i)}{\rho^5},\quad g_{\psi\theta}\sim\frac{f(x^i)}{\rho^4},\quad g_{\theta\theta}\sim\frac{f(x^i)}{\rho^2}$$

$$g_{\varphi\varphi}\sim\frac{f(x^i)}{\rho^2},\quad g_{\varphi\theta}\sim\frac{f(x^i)}{\rho}.$$

The backgrounds corresponding to the m-pole has been discussed in great detail in [43], the interested reader may consult that reference.

6. Summary and discussion.

In the present work it have been found examples of quaternionic kahler spaces with torus symmetry. To find such spaces it have been used a Backglund map allowing to construct eigenfunctions of the hyperbolic laplacian starting with certain solutions of the Ward monopole equations. The advantage of this method is that allows to find non trivial solutions after selecting an arbitrary function of one variable. Examples of G_2 holonomy metrics have been constructed as R^3 bundles over the quaternionic spaces, which asymptotically behave like cones over six dimensional spaces with weak holonomy $SU(3)$. The $U(1) \times U(1)$ isometry of the base space is extended by construction to the total one. This seven dimensional self-dual manifolds have been used to find backgrounds of the eleven dimensional supergravity that gives $N = 1$ supersymmetry after compactification to four dimensions. Such backgrounds are singular for short distances. Type IIA superstring backgrounds have been obtained by reduction along one of the isometries.

Quaternionic manifolds provides a way to construct Spin(7) holonomy manifolds as R^4 bundles over a quaternionic base. The expression of the metric of such manifolds is [6]

$$ds^2 = \frac{dr^2}{\kappa(1 - \frac{c}{r^{10/3}})} + \frac{9}{100\kappa} r^2 (1 - \frac{c}{r^{10/3}})(\sigma^i - A^i)^2 + \frac{9}{20} r^2 ds_4^2$$

and the orbits of constant r (cones) are S^3 bundles over the quaternionic space. This Ricci-flat spaces can be taken as a fundamental space of string theory or a fundamental membrane in M-theory. They can also be used as a domain wall in massive type IIA superstring theory [20].

The hyperkahler and quaternionic kahler geometries are also the typical structures involved by $N = 2$ supersymmetry for the hypermultiplets (see the lectures given in [44]). For this reason another possible physical applications of the Backglund maps presented in this work are in the context of dualities.

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